Indian Statistical Institute, Bangalore
M. Math. Second Year, Second Semester
Advanced Functional Analysis

Final Examination Date: 06-05-2015
Maximum marks: 100 Time: 3 hours

(1) Let X be a topological vector space. Let A, B be non-empty, disjoint, closed subsets of X, where A is compact. Show that there exists an open neighborhood V of 0 such that $(A+V)\bigcap(B+V)=\emptyset$. [15]

(2) Let $\mathcal{B}(Y, W)$ be the space of bounded operators from Y to W, where Y, W are Banach spaces. Let $\{T_n\}_{n\geq 1}$ be a sequence in $\mathcal{B}(Y, W)$, converging in strong operator topology to some bounded operator T. Show that

$$\{\|T_n\| : n \geq 1\}$$

is bounded. [15]

(3) Let $\mathcal{B}$ be a unital commutative Banach algebra. Show that (i) For $x \in B$, $\lambda \in \sigma(x)$ if and only if there exists a complex homomorphism h of $\mathcal{B}$ such that $h(x) = \lambda$. (ii) The Gelfand map is contractive. [20]

(4) Show that every positive linear functional on a unital C^* algebra is continuous. [15]

(5) Let $M_2(\mathbb{C})$ be the C^*-algebra of 2×2 complex matrices with operator norm. Write down a state ϕ on $M_2(\mathbb{C})$, such that $|\phi(A)| = \|A\|$, where

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix}$$

[15]

(6) Let $\mathcal{C}$ be the C^* algebra of $n \times n$ matrices. For a matrix $X = [x_{ij}]$, define $\phi(X) = \frac{1}{n}\sum_{i,j} x_{ij}$. Show that ϕ is a state on $\mathcal{C}$. Describe the GNS triple $(\mathcal{H}, \pi, \xi)$ of ϕ. Compute the dimension of $\mathcal{H}$. [20]

(7) Let $\mathcal{H}$ be a Hilbert space with ortho-normal basis $\{e_n : n \in \mathbb{N}\}$. Let K be a compact subset of $\mathcal{H}$. Show that for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|\sum_{n=N}^{\infty}\langle e_n, x\rangle e_n\| < \epsilon$ for every $x \in K$. [10]

* * **